

Witnessing Continuous Variable Entanglement with Discrete Measurements

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In this Letter, we derive an entropic Einstein-Podolsky-Rosen (EPR) steering inequality for continuous variable (CV) systems using only experimentally measured discrete probability distributions and details of the measurement apparatus. We use this inequality to witness entanglement between the positions and momenta of photon pairs generated in spontaneous parametric downconversion (SPDC). We examine the asymmetry between parties in this inequality, and show that this asymmetry can be used to reduce the technical requirements of experimental setups intended to witness entanglement. Furthermore, we develop a more stringent steering inequality that is symmetric between parties, and use it to witness symmetric EPR steering.

Witnessing entanglement is a task whose difficulty grows with the dimensionality of the system. Witnessing continuous variable (CV) entanglement in the laboratory is understandably challenging because CV systems are infinite-dimensional. As interest in high dimensional entanglement increases, including interest regarding its fundamental aspects [1–11], as well as its use in applications such as quantum key distribution [8], quantum teleportation [12], and quantum computation [9], witnesses of CV entanglement become increasingly important.

Current experimental witnesses of CV entanglement that don't require a complete reconstruction of the density operator or rely on prior knowledge of the system are based either on coarse-grained approximations to the density operator or conditional variance products [6], both of which encounter difficulties at larger dimensions. Since there are as yet no Bell inequalities [13, 14] for CV systems, the strongest level of CV entanglement that can currently be witnessed is EPR steering [15] with the violation of an EPR steering inequality.

EPR steering, first formulated by Wiseman *et al.* [15] requires a level of entanglement weaker than Bell nonlocality [13], but stronger than mere nonseparability [16]. Such steerable states have correlations strong enough to rule out any model of local hidden states (LHS) where conditional distributions of measurement outcomes can be modeled as those arising from single quantum states [17]. All current steering inequalities arise from demonstrations of the EPR paradox [18], when the conditional probability distribution of measurement outcomes of a given particle B, conditioned on measurements of another particle A, is localized to a precision better than the uncertainty principle on particle B allows. Since violating the single-particle uncertainty principle is forbidden for single quantum states, demonstrating the EPR paradox necessarily witnesses EPR steering, which is useful since it allows any formulation of the uncertainty principle to be adapted into an EPR steering inequality [17].

Reid [6] was the first to create an EPR steering inequality using the Heisenberg uncertainty principle. Later,

Walborn *et al.* [2] developed a steering inequality using an entropic [19] formulation of the uncertainty principle for CV position and momentum [20]. For all states whose spatial correlations are insufficient to demonstrate the EPR paradox, the inequality,

$$h(\vec{x}_B|\vec{x}_A) + h(\vec{k}_B|\vec{k}_A) \geq n \log(\pi e), \quad (1)$$

must be satisfied where: n is the number of spatial dimensions; $h(\vec{x}_B|\vec{x}_A)$ is the continuous Shannon entropy [21] of the distribution of measurement outcomes of \vec{x}_B conditioned on outcomes of \vec{x}_A ; $h(\vec{k}_B|\vec{k}_A)$ is similarly defined for measurements of the wavenumber or momentum in natural units; and the base of the logarithm is determined by the units in which we choose to measure the entropy. Walborn *et al.* proved this inequality for one dimension, but it is easily generalized to n dimensions assuming that different spatial degrees of freedom are statistically independent. Since the entropic uncertainty principle is more inclusive than Heisenberg's uncertainty principle [20], the EPR steering inequality derived by Walborn *et al.* is more inclusive than Reid's inequality, as Walborn *et al.* demonstrate with Hermite-Gauss bipartite wavefunctions [2]. Insightful as the entropic steering inequality (1) is, it cannot be used in the laboratory because continuous probability densities cannot be determined with a finite number of measurements.

In this Letter, we derive an exact entropic EPR steering inequality suitable for experimental investigations of CV position-momentum entanglement with discrete measurements:

$$H(\vec{X}_B|\vec{X}_A) + H(\vec{K}_B|\vec{K}_A) \geq \sum_{i=1}^n \log \left(\frac{\pi e}{\Delta x_{Bi} \Delta k_{Bi}} \right). \quad (2)$$

Here we have that $H(\vec{X}_B|\vec{X}_A)$ is the discrete Shannon entropy [21] of measurements of the position of particle B conditioned on measurements of the position of particle A, where both position domains have been discretized into equally spaced windows reflecting the precision of the experimental setup.

Our inequality (2) provides a simpler method to successfully witness EPR steering. We do not need to reconstruct CV probability density functions, and we only require experimentally resolvable discrete probabilities along with those details of the experimental setup needed to determine the measurement resolutions Δx_{Bi} and Δk_{Bi} . Though Reid's inequality is also experimentally useful for variances conditioned on particular measurements, averaging these as a witness for EPR steering may fail to violate their inequality unless all possible conditional variances violate the uncertainty bound. Since our inequality examines the joint probability distribution of measurement outcomes, we cover all possible conditional measurements in a comprehensive way.

To derive our inequality, we will use a fundamental connection between continuous and discrete entropies (8) studied in Ref. [22], to show that any two continuous random variables x and y that can be discretized into equally spaced windows of size Δx and Δy satisfy the following inequality;

$$h(y|x) \leq H(Y|X) + \log(\Delta y). \quad (3)$$

Consider an experiment to measure random variable x which can take the value of any real number with probability density $\rho(x)$. The experiment is only capable of measuring x to discrete windows X_ℓ of size Δx . The probability of measuring x to be in window X_ℓ is

$$P(X_\ell) \equiv \int_{\Delta x_\ell} dx \rho(x), \quad (4)$$

where the region of integration Δx_ℓ is the range of values of x between $x_\ell - \frac{1}{2}\Delta x$ and $x_\ell + \frac{1}{2}\Delta x$, and x_ℓ is the value of x at the center of the window X_ℓ . The Shannon entropy of this discrete probability distribution is given by

$$H(X) = - \sum_\ell P(X_\ell) \log(P(X_\ell)), \quad (5)$$

and the Shannon entropy [21] of the continuous probability density function $\rho(x)$ is expressed as

$$h(x) = - \int dx \rho(x) \log(\rho(x)). \quad (6)$$

We now define the distribution $\rho_\ell(x)$ as the probability distribution of x conditioned on being measured within window X_ℓ . The continuous entropy $h_\ell(x)$ is defined as the entropy of $\rho_\ell(x)$ where

$$\rho_\ell(x) = \frac{\rho(x)}{P(X_\ell)} \quad (7)$$

for all values of x in the window X_ℓ , and is zero otherwise. By breaking up the continuous entropy $h(x)$ into a sum over all windows, and expressing $h(x)$ in terms of

$h_\ell(x)$ and $P(X_\ell)$, we obtain the fundamental connection between discrete and continuous entropies;

$$h(x) = \sum_\ell P(X_\ell) h_\ell(x) + H(X). \quad (8)$$

This connection exists for joint entropies as well as marginal entropies, where we now define $h_{\ell m}(x, y)$ as the entropy of the joint distribution $\rho_{\ell m}(x, y)$ conditioned on x being measured within window X_ℓ and y being measured within window Y_m .

The conditional entropies $h(y|x)$ and $H(Y|X)$ are defined as differences between joint and marginal entropies [21],

$$h(y|x) \equiv h(x, y) - h(x), \quad (9a)$$

$$H(Y|X) \equiv H(X, Y) - H(X). \quad (9b)$$

By using (8) for both single and joint entropies and (9a) and the knowledge that conditioning on additional events reduces the average entropy, it can be shown that

$$h(y|x) \leq \sum_{\ell, m} P(X_\ell, Y_m) h_{\ell m}(y|x) + H(Y|X). \quad (10)$$

The uniform distribution maximizes the entropy [21], so that when all windows Δy_m are of equal size, we have $h_{\ell m}(y|x) \leq \log(\Delta y)$, which completes our proof of (3). Where x_{Ai} and x_{Bi} are another ordinary pair of random variables, we can substitute the expression (3) into the steering inequality created by Walborn *et al.* (1) to derive our entropic EPR steering inequality suitable for experimental investigations of CV entanglement. For a particular spatial degree of freedom $i \in \{1, \dots, n\}$, (i.e. a particular dimension in space) we've shown that

$$H(X_{Bi}|X_{Ai}) + H(K_{Bi}|K_{Ai}) \geq \log \left(\frac{\pi e}{\Delta x_{Bi} \Delta k_{Bi}} \right). \quad (11)$$

When different spatial degrees of freedom are statistically independent of one another, the entropies add, giving us the n -dimensional discrete steering inequality we sought to prove (2).

When applying our steering inequality (2), there are a number of critical details to consider. First, our discrete steering inequalities for each degree of freedom (11) have a cutoff of experimental resolution below which our ability to probe quantum phenomena dependent on the uncertainty principle ceases to exist. For any $\Delta x_{Bi} \Delta k_{Bi}$ larger than πe , we have insufficient resolution to witness entanglement in the i^{th} degree of freedom with our steering inequality (11). In this regime, the steering inequality for the i^{th} degree of freedom (11) becomes impossible to violate since the bound on the right hand side becomes negative, while the sum of discrete entropies on the left hand side is always nonnegative. However, as we increase the resolution, decreasing $\Delta x_{Bi} \Delta k_{Bi}$, we are better able to violate our EPR steering inequality.

Secondly, the inequality (2) appears to depend on the resolutions of only one detector. Though the conditional entropies have an inherent dependence on the resolution of both parties, the inequality is symmetric, as we show in Fig. 1 with experimental data from Ref. [5].

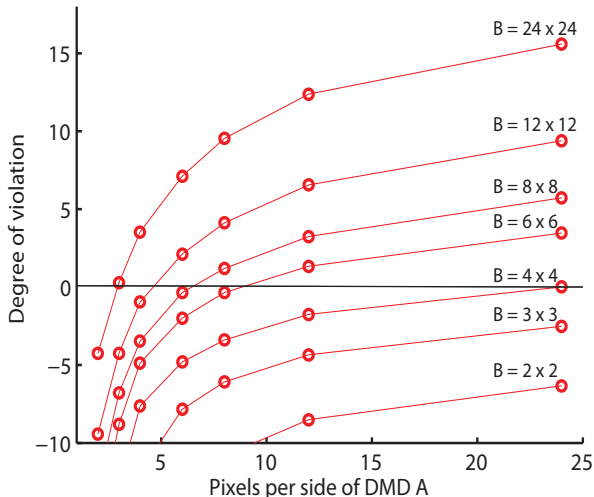


FIG. 1. Violation of our EPR steering inequality (2) for measurements of party B conditioned on measurements of party A as measured in standard deviations of the difference between the bound and the sum of conditional entropies. This plot was formed by independently downsampling the joint distributions for a 24×24 data set. A positive value represents a violation. We see that below 4×4 resolution for party B as suggested by theory, the inequality is not violated for any resolution of DMD A, but violation is possible for large resolution of party B and party A having resolution below 4×4 .

Using Bayes' rule to swap parties [21], we see that symmetry between parties exists only when the marginal entropies for each party are equal to one another. Due to the asymmetry between parties in (2), it is not true in general that the ability to violate the discrete inequality on a given subsystem is simply limited by the party with lowest resolution. However, since the double gaussian wavefunction used to model SPDC is symmetric between parties [5], experimental investigations of EPR steering in SPDC do not exhibit a high degree of asymmetry, though the asymmetry is still significant enough to reduce the necessary technical specifications of an experiment to witness EPR steering. Particularly at higher resolutions, one detector can have less than half the resolution of the other and still be used to witness steering.

In addition, we consider the case where the product of position and momentum windows is fixed, but each is allowed to vary. For large position window sizes: the position conditional entropy decreases toward zero, and the momentum conditional entropy increases without limit. This makes the sum of discrete entropies in (2) grow without limit while the uncertainty bound remains constant,

making EPR steering progressively more difficult to observe. Downsampling the momentum distribution would not make the inequality easier to violate because doing so would also decrease the bound on the right hand side even though it would result in a smaller discrete entropy on the left hand side of the inequality.

Expressing our steering inequality in terms of conditional entropies is useful, but there is also a stronger level of entanglement (which we call symmetric EPR steering) sufficient to allow EPR steering between both parties. This level of entanglement can be witnessed by expressing our inequality in terms of the mutual information. The mutual information [21] is defined as

$$I(\vec{X}_A, \vec{X}_B) \equiv H(\vec{X}_A) + H(\vec{X}_B) - H(\vec{X}_A, \vec{X}_B). \quad (12)$$

Our steering inequality (2) expressed in terms of mutual information becomes

$$I(\vec{X}_A, \vec{X}_B) + I(\vec{K}_A, \vec{K}_B) \leq \sum_{i=1}^n \log \left(\frac{\Delta x_{Bi} \Delta k_{Bi}}{\pi e} \right) + (H(\vec{X}_B) + H(\vec{K}_B)). \quad (13)$$

Since the entropies under discussion are discrete, they are bounded above by the logarithm of the number of windows in the viewing area which can be expressed in terms of ratios $\frac{L_{xi} L_{ki}}{\Delta x_{Bi} \Delta k_{Bi}}$, where L_{xi} and L_{ki} are defined as the total extent of the viewing area in the i^{th} direction for position and momentum measurements of party B, respectively. Canceling out like terms and simplifying, we arrive at a more restrictive steering inequality which is symmetric between parties;

$$I(\vec{X}_A, \vec{X}_B) + I(\vec{K}_A, \vec{K}_B) \leq \log \left(\frac{\prod_{i=1}^n L_{xi} L_{ki}}{(\pi e)^n} \right). \quad (14)$$

Since equation (14) can also be derived by exchanging parties A and B, L_{xi} and L_{ki} can refer to the viewing areas of either party. Violation of this symmetric steering inequality simultaneously witnesses EPR steering for both parties; no conditional distribution of measurement outcomes can be ascribed to a single quantum state.

To use these inequalities in practice, we used data from the experiment in Ref. [5] where we performed measurements of the near field and far field probability distributions (positions and momenta) of pairs of entangled photons generated in SPDC by assembling histograms of coincidence counts. In Ref. [5], we were able to use this joint probability distribution to calculate the conditional entropy and the mutual information, but not to determine whether the system had entanglement. With our new EPR steering inequalities, we can verify that the system in Ref. [5] at certain resolutions does exhibit both EPR and symmetric EPR steering, and therefore is not only entangled, but sufficiently entangled to demonstrate the EPR paradox.

In order to better explain what was measured, we provide a brief description of the experimental setup in Ref. [5] which generated the joint probability distribution that we now analyze with our discrete steering inequalities (2)(14). We separated the downconverted photons with a 50:50 beamsplitter into signal and idler arms, and measured coincident detections using time correlated single photon counting. With these coincidence counts, we measured the joint transverse spatial probability distributions of the photon pairs by imaging the face of the nonlinear crystal (and then its Fourier transform) onto DMD (Digital Micromirror Device) arrays in each arm which allowed us to look at both spatial degrees of freedom in the transverse plane. Uncertainties in the probability distribution were estimated by assuming Poissonian statistics. We measured the joint probability distributions at a variety of different resolutions between 8×8 pixels and 24×24 pixels with the same total viewing area. The time it took to gather a full set of coincidence counts was the primary factor limiting our ability to measure at higher resolutions. At larger resolutions, the number of joint pixel configurations needed to cover the joint probability distribution increases as well as the time it takes to measure a single one of these pixel pairs. Primary sources of error were due to the temperature instability of the nonlinear crystal over the time scales needed to take data, the imperfect alignment of the DMD arrays, and the imperfect in-coupling of light from the DMD arrays into our photodetectors.

With the joint probability distributions obtained from measurements for both the positions and momenta of the photon pairs, we calculated the conditional entropies that go into our steering inequality, and used the details of the experimental setup to determine all the window sizes Δx_{Bi} and Δk_{Bi} . Given the details of the experimental setup, and that the two transverse degrees of freedom are considered independent of one another, the discrete steering inequality (2) takes the form

$$H(\vec{X}_B|\vec{X}_A) + H(\vec{K}_B|\vec{K}_A) \geq 2 \log \left(\frac{\pi e}{\Delta x_B \Delta k_B} \right), \quad (15)$$

where Δk_B is the resolution of the detector of party B to distinguish differences in the horizontal component of the momentum.

The experimental data from Ref. [5] relevant to this inequality is displayed in Fig. 2. In our setup, we were able to violate this inequality by between 3.6 and 16.4 standard deviations for 8×8 resolution to 24×24 resolution respectively. The values of L_x and L_k used to determine Δk_B and Δx_B were $1.04 \times 10^{-3} m$ and $1.00 \times 10^5 m^{-1}$, respectively. The symmetric steering inequality (14) was not violated for 8×8 resolution, but was violated by between 3.4 and 7.0 standard deviations for 16×16 resolution and by between 6.6 and 10.7 standard deviations for 24×24 resolution.

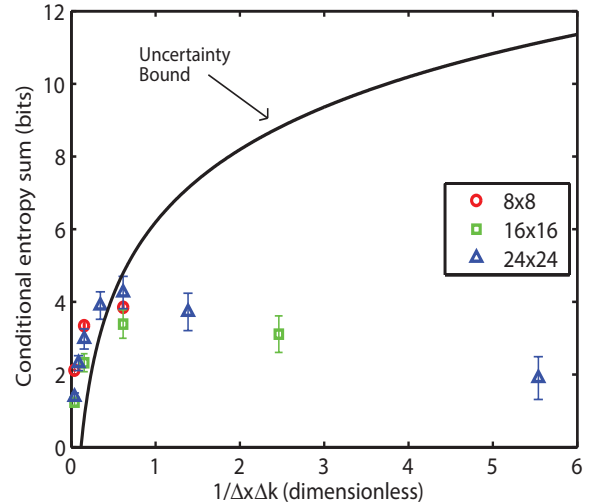


FIG. 2. Experimentally determined sums of conditional entropies in our EPR steering inequality (15) as a function of the product of resolutions $\frac{1}{\Delta x}$ and $\frac{1}{\Delta k}$. The different symbols designate different resolutions of experimental data which were then downsampled. For example, the triangles represent data from an experiment recording at 24×24 resolution. Since the product of the window sizes is dimensionless, this can be expressed in any units we wish. The region below the black curve is the region where our EPR steering inequality (15) is violated; the sum is less than the bound.

In this Letter, we created two new EPR steering inequalities ((2) and (14)) especially well-suited for experimental investigations of CV position-momentum entanglement. Our first inequality is more inclusive and with fewer complications than Reid's inequality [6] at sufficiently high resolution, while both are easier to implement in the lab than the inequality derived by Walborn *et al.* (1). We have successfully witnessed both EPR steering and symmetric EPR steering with experimental data from Ref. [5], and showed that there is a demonstrable asymmetry between parties which allows steering to be witnessed even when one detector has comparatively low resolution provided the other is sufficiently high. These inequalities are powerful and effective characterization tools that we expect to be widely used in applications ranging from future quantum communication networks to fundamental physical experiments involving high-dimensional quantum states.

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